*Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 3, 1998*

# INVERSE PROBLEM OF WING AERODYNAMICS IN A SUPERSONIC FLOW

N. F. Vorob'ev

UDC 533.69

*The inverse problem of wing aerodynamics — the determination of the lifting surface shape from a specified load — is solved within the framework of linear theory. Volterra's solution of the wave equation is used. Solutions are found in the class of bounded functions if certain conditions imposed on the governing parameters of the problem are satisfied. Solutions of inverse problems of supersonic flow are presented for an infinite-span wing, a triangular wing with completely subsonic edges, and a rectangular wing.*

The problems of inviscid supersonic flow around a thin slightly curved finite-span wing in a linear formulation reduce to solving the wave equation for the velocity potential with time-oriented data on the base plane. The Volterra representation of the solution of the wave equation allows one to choose either the normal derivative — the wing geometry (the direct aerodynamic problem) or the load function on the wing (the inverse aerodynamic problem) as the governing parameter on the base plane [1].

The solution of the inverse problem is presented as the potential

$$\Phi(x,y,z) = \frac{y}{\pi}\iint_s \frac{\Phi'_\xi(\xi,\zeta)(x-\xi)}{[(z-\zeta)^2+y^2]\sqrt{(x-\xi)^2-[(z-\zeta)^2+y^2]}}\,ds. \tag{1}$$

Here $s$ is the domain of influence of the point $M(x,y,z)$ on the base plane $\eta = 0$ and $\Phi'_\xi(\xi,\zeta)$ is the pressure difference on the plane $\eta = 0$.

The velocity potential in the direct and inverse problems is written as double integrals whose integrands (the kernels of the integral operators) contain singularities. In finding the gas-dynamic flow parameters (the velocity-potential derivatives), the power of the integrand singularities increases, and, sometimes it is impossible to perform formal differentiation operations within the framework of bounded functions. In some cases, differentiation gives rise to singularities that make the integrals divergent. The method of recognizing the existence of integrals in the sense of Hadamard is often used [2]. The introduction of such symbols not only complicates implementation of the algorithms of solution but, sometimes, requires justification of physically absurd results. The observance of the rules of differentiation of integrals with variable limits and the requirement of necessary smoothness on the wing surface allow one to obtain gas-dynamic parameters of the flow in the class of bounded functions [3, 4].

Based on the representation of the solution of the inverse problem of wing aerodynamics as potential (1), the dependence of the normal derivative $\Phi'_y$ (the wing geometry) on the derivative $\Phi'_x$ (the load on the wing) is determined from the formula [1, 4]

$$\Phi'_y\Big|_{y=0} = -\Phi'_x\Big|_{y=0} + \frac{1}{\pi}\int_{COD}\left[\Phi'_\xi\,\frac{\sqrt{(x-\xi)^2-(z-\zeta)^2}}{(x-\xi)(z-\zeta)}\right]_{\zeta=f(\xi)} d\xi + \frac{1}{\pi}\iint_s \Phi''_{\xi\zeta}\,\frac{\sqrt{(x-\xi)^2-(z-\zeta)^2}}{(x-\xi)(z-\zeta)}\,d\zeta\,d\xi. \tag{2}$$

Here $s(MCAOBDM)$ is the domain of influence of the point $M(x,0,z)$ on the base plane $\eta = 0$, $\zeta = f(\xi)$ is the equation of the line $COD$, which is the boundary between the domain $s$ and the free stream (Fig. 1),

Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 3, pp. 86–91, May–June, 1998. Original article submitted September 30, 1996.

 

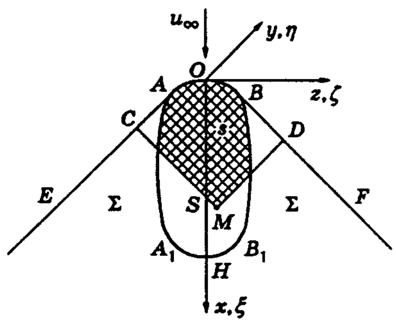


Fig. 1

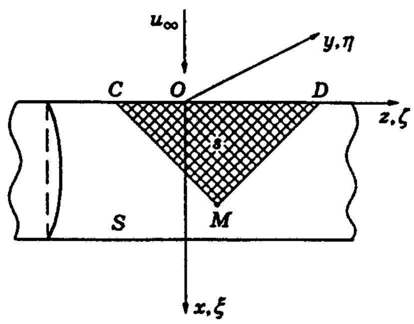


Fig. 2

and $\Phi''_{\xi\zeta}$ is the derivative of the load along the $\zeta$ axis.

Formula (2) yields a solution of the inverse problem in quadratures if the governing parameters $\Phi'_\xi$ and $\Phi''_{\xi\zeta}$ are specified over the entire disturbed region on the plane $\eta = 0$. In the region of the wing projection $S$ and on its boundary $L$ (including the section $L_0$, at the supersonic leading edge $AOB$), $\Phi'_\xi = p(\xi, \zeta)$ and $\Phi''_{\xi\zeta} = p'_\zeta(\xi, \zeta)$ should be defined according to the formulation of the inverse problem. In the region $\Sigma$ (a part of the disturbed region on the plane $\eta = 0$ outside $S$) and on its boundary with the free stream (on the front characteristics $AE$ and $BF$), we have $\Phi'_\xi = \Phi''_{\xi\zeta} = 0$. Thus, in formula (2), the integration is performed only over the part $s \in S$ (the dashed region in Fig. 1), and in the contour integral, the integration is performed only over the section $L_0$.

The kernels of the integral operators in (2) have singularities of the type of $\lim\limits_{\xi \to x}(1/(x - \xi))$. To obtain a solution of the inverse problem in the class of bounded functions, it is necessary to impose the following conditions on the governing parameters $p(\xi, \zeta)$ and $p'_\zeta(\xi, \zeta)$ specified in the region $S + L$ [1, 4].

(1) Continuity of the load $p(\xi, \zeta)$ over the entire disturbed region $S + L$. Since $p(\xi, \zeta) = 0$ in the region $\Sigma$, the condition $p(\xi, \zeta) = 0$ should be satisfied on the subsonic part of the wing leading edge $L_1$ (the curves $AA_1$ and $BB_1$).

(2) At the trailing edge of the wing ($A_1HB_1$), we have $p(\xi, \zeta) = 0$ if it is subsonic, and no conditions are set for $p(\xi, \zeta)$ if it is supersonic.

(3) Continuity of the derivative $p'_\zeta(\xi, \zeta)$ in the region $S$ and its boundedness on $L$.

After satisfaction of these necessary conditions for the existence of integrals in the class of bounded functions, a problem arises for their representation in elementary functions (the integrals should be reducible to tabulated integrals). This makes one specify the governing parameter of the problem $p = p(\xi, \zeta)$ in the simplest form.

The objective of the inverse problem for a wing of specified planform is to determine the wing surface shape. According to the linearized nonpenetration condition $\Phi'_y = u_\infty \sin\alpha \approx u_\infty \tan\alpha$, where $\alpha = \alpha(x, z)$ is the slope of the tangent line to the $x$ axis in the cross section $z = \text{const}$, we have $dy/dx = \tan\alpha$ for the point $M(x, 0, z) \in S$. The derivative $\Phi'_y$ is given by the relation $\Phi'_y = F(x, z)$, where $F(x, z)$ is the right-hand side of formula (2). Thus, the equation of the surface $S$ is

$$y(x, z) = \frac{1}{u_\infty} \int\limits_{x_0}^{x} F(\xi, z)\, d\xi. \tag{3}$$

Here $x_0 = f^-(z)$ is the leading-edge coordinate in the cross section $z = \text{const}$.

We give some examples of solution of the inverse problem (2) for wings of simple plan form.

(1) The first test case is the solution for a wing of infinite span in the $z$ direction (Fig. 2). In this case, the domain of influence of the point $M$ lies entirely on the wing projection $S$, where the load $\Phi'_\xi = p(\xi)$ is specified. In formula (2), the contour integral along the curve $COD$ disappears ($\xi = 0$), and the double integral vanishes because the flow is plane–parallel ($\Phi''_{\xi\zeta} = 0$). Thus, we obtain the following known relation

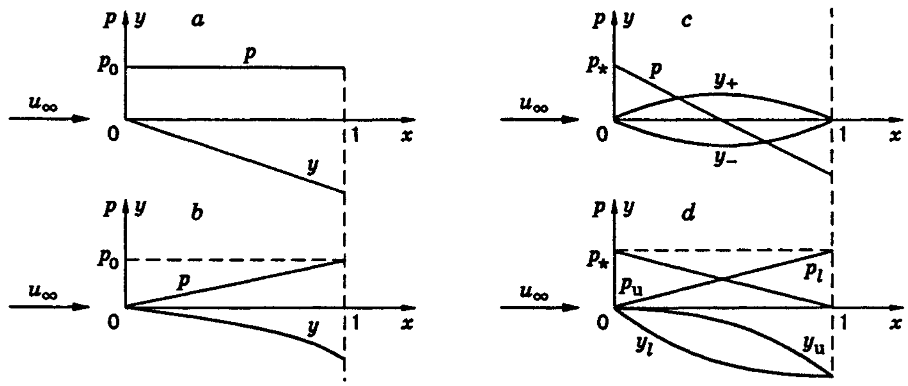


Fig. 3

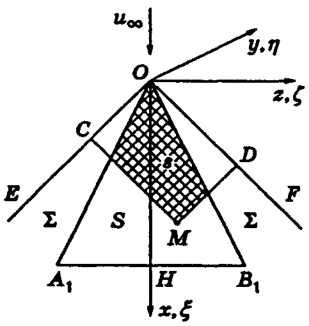


Fig. 4

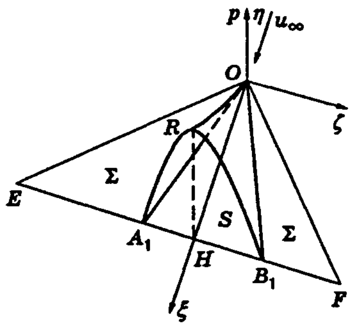


Fig. 5

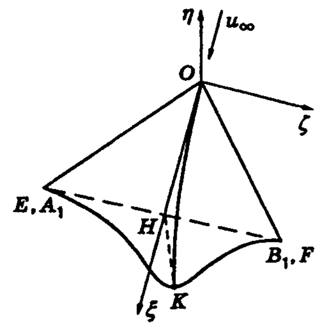


Fig. 6

for an infinite-span wing:

$$\Phi'_y\Big|_{y=0} = -\Phi'_x\Big|_{y=0} = -p(\xi). \tag{4}$$

In the case of an infinite-span wing, the erroneous (see [4])[1] solution of the inverse problem [2] used in the literature,

$$\Phi'_y\Big|_{y=0} = -\Phi'_x\Big|_{y=0} + \frac{1}{\pi}\iint\limits_{s} \Phi'_\xi \frac{(x-\xi)}{(z-\zeta)^2\sqrt{(x-\xi)^2-(z-\zeta)^2}}\,d\zeta\,d\xi,$$

does not transform to the above relation (4). A double integral which should be understood in the sense of Hadamard is retained.

In accordance with (2)–(4), the equation of the lifting line is

$$y = -\frac{1}{u_\infty}\int_0^x p(\xi)\,d\xi, \qquad 0 \leqslant x \leqslant 1. \tag{5}$$

Figure 3 plots the lifting lines $y = y(x)$ for various types of load obtained using formula (5): (a) constant load $p = p_0$ and the lifting line is the plate $y = -(p_0/u_\infty)\,x$; (b) linearly increasing load with a shockless entrance $p = p_0 x$ and the lifting line is $y = -(p_0/u_\infty)x^2/2$; (c) load $p = p_*(1-2x)$ and the lifting lines are arcs that form the parabolic profile $y = \pm(p_*/u_\infty)x(1-x)$; (d) the linearly increasing $p = p_* x$ and linearly decreasing $p_l = p_*(1-x)$ loads correspond to the arcs $y_u = -(p_*/u_\infty)x^2/2$ and $y_l = -(p_*/u_\infty)(x - x^2/2)$ which form a closed profile with constant pressure difference $p_l + p_u = p_*$ over the entire length of this profile.

(2) A triangular wing with entirely subsonic leading edges (Fig. 4) is loaded according to the law

$$p = \frac{p_0}{k^2}(k^2\xi^2 - \zeta^2), \quad p'_\zeta = -\frac{2p_0}{k^2}\zeta, \quad (\xi,\zeta) \in S + L, \quad 0 < k \leqslant 1. \tag{6}$$

[1]The notation of the axes $(y,\eta)$ and $(z,\zeta)$ in the figure of [4] should be interchanged.

Figure 5 shows schematically the surface $p = p(\xi, \zeta)$ $(OA_1RB_1O)$. The region $S$ (the triangle $OA_1B_1O$), which is the wing projection onto the base plane $\eta = 0$, is bounded by the contour $L$: the leading edges of the projection are $OA_1$ and $OB_1$ $(\zeta = \mp k\xi)$ and the trailing edge is $A_1B_1$ $(\xi = \text{const})$. There is no contour integral in formula (2) for such a wing, since $p = 0$ for the leading characteristics $OE$ and $OF$ $(\zeta = \mp\xi)$, according to the formulation of the problem. In the double integral, according to the formulation of the problem, the integration is performed only over the region $S$. Having performed all integration procedures imposed by formulas (2) and (3), we can write the equation for the surface $y = y(x, z)$ of the triangular wing $OA_1B_1O$ loaded according to formula (6):

$$y = -\frac{2\,p_0}{3\,k^2}\,x^3 G(c, k), \tag{7}$$

where

$$G(c,k) = (k^2 - c^2) - \frac{1}{\pi}\left\{k\sqrt{1-c^2} - 2\,\frac{c^2}{k}\ln\frac{1+\sqrt{1-c^2}}{c}\right.$$

$$+\frac{(k-c)[k^2(k-c) - 2(1-k^2)c]}{2k\sqrt{1-k^2}}\ln\frac{\sqrt{(1-k^2)(1-c^2)} + 1 - kc}{(k-c)}$$

$$\left.+\frac{(k+c)[k^2(k+c) + 2(1-k^2)c]}{2k\sqrt{1-k^2}}\ln\frac{\sqrt{(1-k^2)(1-c^2)} + 1 + kc}{(k+c)}\right\}, \quad 0 < c \leqslant k \leqslant 1,$$

$c = z/x$ is the ray in the base plane $y = 0$, and $k = z/x$ is a ray that corresponds to the wing edge.

The equation of the lateral edge of the wing $(c = k)$ is

$$y = \frac{2\,p_0}{3\,k\pi}\,x^3\left\{\sqrt{1-k^2} - 2\ln\frac{1+\sqrt{1-k^2}}{k} - \frac{2}{\sqrt{1-k^2}}\ln k\right\}.$$

The equation of the centerline $(c = 0)$ is

$$y = -\frac{2\,p_0}{3\,k}\,x^3\left\{k - \frac{1}{\pi}\left[1 + \frac{k^2}{\sqrt{1-k^2}}\ln\frac{1+\sqrt{1-k^2}}{k}\right]\right\}.$$

In the case of the wing with subsonic edges $(k = 1)$, the equation of the wing surface (7) takes the form

$$y = -\frac{2\,p_0}{3}\,x^3\left\{(1-c^2) - \frac{2}{\pi}\left[\sqrt{1-c^2} - c^2\ln\frac{1+\sqrt{1-c^2}}{c}\right]\right\}.$$

For the leading edges $OA_1$ and $OB_1$ $(c = k = 1)$, we have $y = 0$, and for the centerline $OK$ $(c = 0,\ k = 1)$, we obtain

$$y = -\frac{2\,p_0}{3}\,x^3\left(1 - \frac{2}{\pi}\right).$$

The wing supports the base plane $y = 0$ by the straight line edges $OA_1$ and $OB_1$ (Fig. 6) and enters the flow, introducing no disturbances by the leading edges.

(3) A rectangular wing (Fig. 7a) is uniformly loaded in its root part, and the load on the wing panels ensures the required smoothness in conjugating loads in the regions $S$ and $\Sigma$ to obtain a solution in the class of bounded functions (Fig. 7b):

$$S:\ \ \Phi'_\xi = p_0, \qquad \Phi''_{\xi\zeta} = 0, \qquad 0 \leqslant \zeta \leqslant a,$$

$$\left.\begin{aligned} \Phi'_\xi &= \frac{p_0}{(b-a)^3}(b-\zeta)^2[3(\zeta-a) + (b-\zeta)] \\ \Phi''_{\xi\zeta} &= -\frac{6\,p_0}{(b-a)^3}(b-\zeta)(\zeta-a) \end{aligned}\right\} a \leqslant \zeta \leqslant b;$$

$$\Sigma:\ \ \Phi'_\xi = \Phi''_{\xi\zeta} = 0, \qquad \Gamma_0:\ \ \Phi'_\xi = 0.$$

Here $\Gamma_0$ is the leading characteristic line (Fig. 7a).

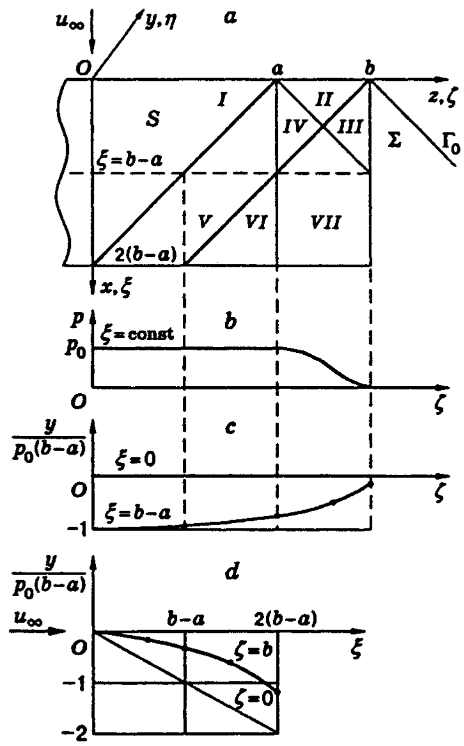


Fig. 7

For a wing with chord $2(b - a)$, where $(b - a)$ is the spanwise length of the wing panel, there are seven domains (domains I–VII are separated by solid lines) with different types of analytical solution, which depends on the wing geometry and load distribution over the wing. The integration of singular integrals according to formulas (2) and (3), which is necessary to obtain an analytical equation of the wing surface, is a cumbersome procedure that requires special attention to singularities of the operators. The wing-surface equations obtained for domains I–VII reveal the characteristic features of the wing geometry, but they are not presented here because they are too cumbersome.

Figure 7c and d shows the wing surface behavior in characteristic cross sections. The leading edge ($\xi = 0$) remains rectilinear on the wing panel ($a \leqslant \zeta \leqslant b$). In the cross section $\xi = b - a$, the wing is "adjusted" to the pressure distribution specified on the wing panel, beginning from the point of intersection with the characteristic that emerges from the point ($\xi = 0$, $\zeta = a$). The root cross section ($(\zeta = 0)$), where the pressure is constant ($p = p_0$) and the influence of the wing panel is absent, is a straight line (the wing surface in domain I is a plate). At the end face ($\zeta = b$), the wing is "inscribed" into the free stream and does not introduce disturbances by the corner point ($\xi = 0$, $\zeta = b$).

## REFERENCES


1. N. F. Vorob'ev, *Lifting-Surface Aerodynamics in a Steady Flow* [in Russian], Nauka, Novosibirsk (1985).
2. W. R. Sears (ed.), *General Theory of High-Speed Aerodynamics*, Princeton Univ. Press, New Jersey (1957).
3. N. F. Vorob'ev, "An exact solution of the end-effect problem for a wing of finite span in a supersonic flow," *Prikl. Mekh. Tekh. Fiz.*, **33**, No. 1, 65–71 (1992).
4. N. F. Vorob'ev, "Unique features of the aerodynamics problems for a wing of finite span," *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 3, 55–66 (1994).